Eur. Phys. J. B **5**, 647–661 (1998)

THE EUROPEAN PHYSICAL JOURNAL B

Elliptic curves from finite order recursions or non-involutive permutations for discrete dynamical systems and lattice statistical mechanics*

N. Abarenkova[1,a], J.-C. Anglès d'Auriac[1,b], S. Boukraa[2,c], and J.-M. Maillard[2,d]

[1] Centre de Recherches sur les Très Basses Températures, BP 166, 38042 Grenoble, France
[2] LPTHE, tour 16, 4 place Jussieu, 75252 Paris Cedex, France

Received: 9 February 1998 / Revised: 13 March 1998 / Accepted: 17 March 1998

Abstract. We study birational mappings generated by matrix inversion and permutations of the entries of $q \times q$ matrices. For $q = 3$ we have performed a systematic examination of all the birational mappings associated with permutations of 3×3 matrices in order to find integrable mappings and some finite order recursions. This exhaustive analysis gives, among 30 462 classes of mappings, 20 classes of integrable birational mappings, 8 classes associated with integrable recursions and 44 classes yielding *finite order* recursions. An exhaustive analysis (with a constraint on the diagonal entries) has also been performed for 4×4 matrices: we have found 880 new classes of mappings associated with integrable recursions. We have visualized the orbits of the birational mappings corresponding to these 880 classes. Most correspond to elliptic curves and very few to surfaces or higher dimensional algebraic varieties. All these new examples show that integrability can actually *correspond to non-involutive permutations*. The analysis of the integrable cases specific of a particular size of the matrix and a careful examination of the non-involutive permutations, shed some light on the integrability of such birational mappings.

PACS. 05.50.+q Lattice theory and statistics; Ising problems – 02.10.-v Logic, set theory, and algebra – 02.20.-a Group theory

1 Introduction

Birational transformations naturally pop out as non trivial symmetries of lattice models of statistical mechanics and solid state physics. For example a set of (birational) transformations of the R-matrix of the sixteen-vertex model [1] exits which are non trivial symmetries of the parameter space of the model. These transformations find their origin in the so-called inversion relation [2] and in the lattice symmetries. They form a (generically infinite discrete) group generated by the composition of two involutions. A worth noticing property of integrability has been found for this generator, opening the question whether this integrability property is related to an underlying statistical mechanics model or not. To answer this question a wide class of birational mapping has been introduced moving the point of view from statistical mechanics to discrete dynamical system.

In previous papers birational mappings [3–5] having their origin in the theory of exactly solvable models in lattice statistical mechanics [6–9, 12, 13] have been first studied. They are generated by *involutive transformations on matrices* corresponding to two kinds of transformations on $q \times q$ matrices: the inversion of the $q \times q$ matrix and an (*involutive*) *permutation of the entries* of the matrix. In these papers, *permutations of two entries* [3–5], as well as permutations corresponding to *discrete symmetries* of lattice models of statistical mechanics [6–9, 12, 13] were first analysed. Several integrable mappings associated with permutations of $q \times q$ matrices, *for arbitrary* q, have been found this way [3–5]. It has also been shown that the iteration of the associated birational transformations presents some remarkable *factorization properties* [3, 4]. These factorization properties explain why the complexity of these iterations, instead of having the exponential growth one expects at first sight, may have a *polynomial growth* of the complexity [3, 4, 15, 16]. It has also been shown that the polynomial factors occurring in these factorizations *may satisfy noteworthy non-linear recursion relations* and that some of these recursions were actually *integrable*, yielding elliptic curves [3, 4].

We perform here a systematic examination of such birational mappings associated with *all* the permutations

* Dedicated to J. Zittartz on the occasion of his 60th birthday

[a] Theoretical Physics Department, Sankt Petersburg State University, Ulyanovskaya 1, 198904 Sankt Petersburg, Russia.
[b] e-mail: dauriac@crtbt.polycnrs-gre.fr
[c] Institut d'Aéronautique, Université de Blida, BP 270, Blida, Algérie.
[d] e-mail: maillard@lpthe.jussieu.fr

of entries of 3×3 matrices and (almost all) permutations of entries of 4×4 matrices as well. This analysis provides a set of new integrable mappings of various number of (homogeneous) variables (3^2, 4^2, arbitrary number). Our motivation is not only to accumulate as many new integrable mappings as possible, but rather to answer the two following questions. Firstly, is the integrability of these birational mappings *necessarily associated with involutive* permutations? Secondly, in which context higher dimension Abelian varieties, rather than elliptic curves, occur?

The paper is organized as follows: in the next section we give our notations and recall some known results. The seminal case of the sixteen-vertex model is presented, together with possible generalizations, which give the motivation to study more general birational mappings associated to permutations of q^2 elements. In the third section, an exhaustive study of all the mappings associated to permutations of 3^2 elements is presented. This includes the case of *non involutive* permutations. Completely integrable cases or completely chaotic cases are found, but also intermediary cases. In the fourth section we exemplify the generic situations encountered with specific examples that we study in details. The fifth section is devoted to the study of permutations of 4^2 elements. This study is almost exhaustive and we give explicitly the integrable mappings we have found. We will stress that integrable mappings are extremely seldom. Moreover they do not always correspond to involutive permutations suggesting that this very integrability could not be related to an underlying statistical mechanics model. We conclude with a section devoted to summarize our main results and to state some questions which seem interesting to us.

2 Recalls

We first introduce some notations. $\mathcal{S}_{q\times q}$ denotes the set of permutations of the q^2 entries of a $q\times q$ matrix. The $q\times q$ matrix $\mathcal{M}_n$ is obtained iterating n times an arbitrary initial homogeneous matrix M_0:

$$\mathcal{M}_n \;=\; \widehat{K}_t^n(M_0) \tag{1}$$

where $\widehat{K}_t = t\cdot\widehat{I}$, $\widehat{I}(M) = M^{-1}$ and $t \in \mathcal{S}_{q\times q}$. Transformation $\widehat{K}_t$ is clearly a *birational transformation* on the entries of M_0 since its inverse, $\widehat{I}t^{-1}$, is also a rational transformation. The entries of the matrix are defined up to an arbitrary multiplicative constant. So we also define the homogeneous transformation $K_t = tI$, where $I(M) = \det(M)M^{-1}$ (homogeneous inverse). Obviously K_t and $\widehat{K}_t$ yield the same iteration in the projective space $\mathrm{CP}_{q^2_{-1}}$. From a numerical point of view the iteration of $\widehat{K}_t$, seen as a discrete dynamical system, is more convenient, whereas K_t is easier to handle for analytical calculations. Since all the entries of $K_t^n(M)$ are homogeneous polynomials in q^2 variables we will also use the 'reduced' matrices M_n which are obtained dividing all entries of $K_t(M_{n-1})$ by their common factor (GCD). It also often happens that the determinant of the reduced matrices factorizes itself.

Let us illustrate this with the example of the mapping corresponding to the sixteen vertex models [13,16]. We introduce the permutation t_1 which exchanges[1] the two 2×2 off diagonal sub-matrices of a 4×4 R-matrix. The mapping $K_{t_1} = t_1 I$ corresponds to some non trivial *non linear* symmetry of the (4×4) R-matrix of the *sixteen vertex* [1] model[2]. In that precise case a sequence of factorization occurs, which we call a factorization scheme. It reads:

$$M_{n+2} \;=\; \frac{K_{t_1}(M_{n+1})}{f_n^2}, \qquad f_{n+2} \;=\; \frac{\det(M_{n+1})}{f_n^3},$$
$$\widehat{K}_{t_1}(M_{n+2}) \;=\; \frac{K_{t_1}(M_{n+2})}{\det(M_{n+2})} = \frac{M_{n+3}}{f_{n+1}f_{n+3}}\cdot \tag{2}$$

One has a *hierarchy of recursions integrable*, or compatible with integrability [16]:

$$\frac{f_n\, f_{n+3}^2 \;-\; f_{n+4}\, f_{n+1}^2}{f_{n-1}\, f_{n+3}\; f_{n+4} \;-\; f_n\, f_{n+1}\, f_{n+5}} \;=\; \frac{f_{n+1}\, f_{n+4}^2 \;-\; f_{n+5}\, f_{n+2}^2}{f_n\, f_{n+4}\; f_{n+5} \;-\; f_{n+1}\, f_{n+2}\, f_{n+6}}\cdot \tag{3}$$

Defining the variables l_n and x_n:

$$l_n \;=\; \det(\widehat{K}_t^n(M_0)) \quad \text{and} \quad x_n \;=\; l_n l_{n+1} \tag{4}$$

one has the simplest recursion on these new variables:

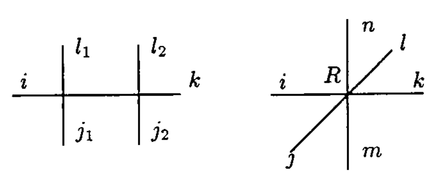
$$\frac{x_{n+2}-1}{x_{n+1}\,x_{n+2}\,x_{n+3}-1} = \frac{x_{n+1}-1}{x_n\,x_{n+1}\,x_{n+2}-1}\,x_n\,x_{n+1}\,x_{n+2}^2\cdot \tag{5}$$

This recursion has been integrated in [16], where the foliation of the 16-dimensional parameter space is also given.

Another example of permutation, which originates from symmetry analysis of vertex models, generalizes the previous permutation t_1 and corresponds to the following action on a $2m\times 2m$ R-matrix [13,16,18]:

$$t_1\!: \quad R \;=\; \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \longrightarrow \quad t_1(R) \;=\; \begin{pmatrix} A & C \\ B & D \end{pmatrix} \tag{6}$$

where A, B, C and D are $m\times m$ matrices. This last permutation t_1 corresponds to the analysis of vertex models on a *cubic* (or d-dimensional hypercubic) lattice [16,18] ($m = 4$, $m = 2^{d-1}$), as well as *monodromy matrices* of vertex models on a square lattice [16] ($m = 2^N$, N number of sites in the monodromy matrix).



(7)

[1] Transformation t_1 is a geometrical symmetry of the square lattice [13].

[2] The Baxter model is a Yang-Baxter integrable subcase of this model [13]. One should not confuse the integrability of the *symmetries* of the parameter space of the sixteen vertex model (namely the mappings considered here) and the Yang-Baxter integrability [13]: the *sixteen vertex model is not generically Yang-Baxter integrable.*

The factorization scheme reads (with $q = 2m$ and using the same notations):

$$K(M_n) = M_{n+1} f_n^{q-5} f_{n-1}^5 f_{n-2}^{2(q-5)} f_{n-3}^6 f_{n-4}^{2(q-5)} f_{n-5}^6 \cdots$$
$$\det(M_n) = f_{n+1} f_n^{q-4} f_{n-1}^7 f_{n-2}^{2(q-4)} f_{n-3}^8 f_{n-4}^{2(q-4)} f_{n-5}^8 f_{n-6}^{2(q-4)} \cdots \quad (8)$$

It has been seen [16,18] that permutation (6) yields a *polynomial growth* of the calculations, however the f_n's, or the x_n's, *do not* verify any recursion relation like (3) or (5). In fact the orbits of the associated mappings K_{t_1} can be seen [16,18] to (uniformly) densify an (Abelian) algebraic variety of dimension g which depends on m. In principle one can explicitly write the evolution of the points in term of theta functions of g variables. One could call such a situation a "g-integrability". Furthermore it has been seen that an R-matrix of cubic vertex model (or of a triangular vertex model, 32-vertex model [16]) can, after some rearrangement of lines and columns, be written as the direct sum of two 4×4 identical matrices, the matrix inversion inherited on these 4×4 matrices, being the matrix inversion of the 4×4 matrices, but permutation of entries t_1 becoming a quite complicated permutation of the entries of these 4×4 matrices [16].

This fully justifies to consider the following problem combining for 4×4 matrices (resp. $q \times q$) the matrix inversion with quite general arbitrary permutation of the entries of the 4×4 (resp. $q \times q$) matrices. For instance, it would be interesting to find exhaustively, for 3×3 matrices, all the permutations yielding such "g-integrability" (or more simply yielding *polynomial growth* [16,18]). Unfortunately we do not have any simple criterion, or any quick and efficient algorithm, to perform such an exhaustive search. Therefore we restrict, in this paper, to the "traditional" integrability (foliation of the parameter space in elliptic, or rational, curves) and to "g-integrability" of birational mappings such that the previous determinantal variables x_n's satisfy specific remarkable integrable recursions.

In this context simple permutations of the entries of $q \times q$ matrices, like transposition of two entries, have been analyzed [3–5]. Let us recall here a few simple results. Let us denote $\mathcal{S}^2_{q\times q}$ the subset of $\mathcal{S}_{q\times q}$ which consists in a simple interchange of *two* entries.

Let us introduce the permutations of entries $g_{i,j}$ $(0 \le i < j < q)$ which consist in the exchange of column i with column j followed by the exchange of row i with row j. It is straightforward to see (with obvious notations) that:

$$K_{g^{-1}tg}(M_0) \;=\; g^{-1} K_t(M_0) g. \quad (9)$$

This means that the equivalence relation, defined by $t' = g^{-1}tg$, is compatible with iteration (1). From now on K_t will simply be denoted K. Up to these row and columns *relabeling* equivalence relation, one can show that $\mathcal{S}^2_{q\times q}$ yields *only six classes* [3]: iteration (1) has to be analyzed for a only one representative in each class. The class denoted class I in [3] can be represented by the interchange of the two entries $M_0[1,2]$ and $M_0[2,1]$. Class I presents remarkable factorization properties at each step of iteration (1). The determinants of the iterated matrices do factorize and all the entries of an iterated matrix also factorize a common (homogeneous) polynomial. Thus, at the n-th step of the iteration, one can introduce the "reduced matrices" M_n's and introduce homogeneous polynomials denoted f_n's corresponding to these very factorizations. For class I iteration (1) thus yields:

$$M_{n+1} = \frac{K(M_n)}{f_{n-2}^{q-2}\, f_{n-1}^2\, f_n^{q-4}}$$
$$f_{n+1} = \frac{\det(M_n)}{f_{n-2}^{q-1}\, f_{n-1}^3\, f_n^{q-3}} \quad (10)$$

with $f_n = 1$ for $n \le 0$. Moreover the degree of the f_n's grows *polynomially* with n (quadratic growth [3]). Remarkably, these homogeneous polynomials f_n's do satisfy a *whole hierarchy of non-linear recursion relations* [3,4] *independent* of q. A simple recursion in this hierarchy is the (integrable) recursion:

$$\frac{f_n f_{n+3}^2 - f_{n+4} f_{n+1}^2}{f_{n-1} f_{n+3} f_{n+4} - f_n f_{n+1} f_{n+5}} = \frac{f_{n-1} f_{n+2}^2 - f_{n+3} f_n^2}{f_{n-2} f_{n+2} f_{n+3} - f_{n-1} f_n f_{n+4}} \cdot \quad (11)$$

The K-orbit of an arbitrary matrix M_0 is an *elliptic curve* in the parameter space of the q^2 homogeneous entries of M_0: the mapping K itself is *integrable* [3] for *arbitrary values of* q. These two integrability properties are of course related [3].

The x_n's variables (see (4)) also verify a *whole hierarchy of non-linear recursion relations* [3,4] closely related to the existence of the recursions on the f_n's [3,4]. Let us give the simplest recursion of this hierarchy:

$$\frac{x_{n+2} - 1}{x_{n+1}\, x_{n+2}\, x_{n+3} - 1} = \frac{x_{n+1} - 1}{x_n\, x_{n+1}\, x_{n+2} - 1} x_n\, x_{n+2} \cdot \quad (12)$$

This recursion is *integrable*. In the following, condition (12) is used as an *integrability criterion* ("class I-integrability").

Recursion (12) has been integrated in [3] and yields *biquadratic relations* in terms of some new (homogeneous) variables q_n defined by $x_n = q_{n+1}/q_n$:

$$(\rho - q_n - q_{n+1})\,(q_n q_{n+1} + \lambda) = \mu. \quad (13)$$

The x_n's and q_n's are related to ratio of the polynomials f_n's:

$$x_n = \frac{f_{n-1}^2\, f_{n+2}}{f_{n+1}^2\, f_{n-2}}, \qquad q_n = \frac{f_{n-2}\, f_{n+1}}{f_{n-1}\, f_n} \cdot \quad (14)$$

Studying the iteration of $\widehat{K}^2$ in the (3^2-1)-dimensional projective space of the entries of M_0, $\mathbf{CP}_{3^2-1} = \mathbf{CP}_8$, one can show that these orbits correspond to *elliptic* *curves* and actually belong to a *three-dimensional* vector-space:

$$\widehat{K}^{2\,n}(M_0)=a_0 M_0+a_1\widehat{K}^2(M_0)+a_2\widehat{K}^4(M_0)+a_3\widehat{K}^6(M_0). \tag{15}$$

One can also see that the following constant matrix always belongs to this three-dimensional vector-space:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{16}$$

Therefore, in the right-hand side of (15), $\widehat{K}^6(M_0)$ can be replaced by the constant matrix P, the three-dimensional vector-space yielding, with homogeneous notations:

$$K^{2\,n}(M_0) = a_0 M_0 + a_1 K^2(M_0) + a_2 K^4(M_0) + a_3 P. \tag{17}$$

These results are specific of $q = 3$. Let us note that this three-dimensional vector-space becomes [4] for $q \geq 4$, a five dimensional vector-space:

$$\begin{aligned}\widehat{K}^{2\,n}(M_0) = a_0^{(n)} M_0 \; + \; a_1^{(n)}\widehat{K}^2(M_0) \; + \; a_2^{(n)}\widehat{K}^4(M_0) \\ + a_3^{(n)}\widehat{K}^6(M_0) \; + \; a_4^{(n)}\widehat{K}^8(M_0) \; + \; a_5^{(n)} P.\end{aligned} \tag{18}$$

One can show, for $q \geq 3$, that matrix P is unique. One can in fact (see Appendix) represent transformation $\widehat{K}^2$ or the homogeneous transformation K^2, as (birational) mappings on four (resp. six) homogeneous coordinates $a_0^{(n)}$, $a_1^{(n)}$, $\cdots$, $a_3^{(n)}$ (resp. $a_5^{(n)}$). The "price to pay" is that this representation of $\widehat{K}^2$ (or K^2) depends in a very complicated way of the initial matrix M_0. Not surprisingly these transformations are integrable yielding elliptic curves. This is thus a way to get an *infinite number* of new three-dimensional, or five-dimensional, integrable mappings (see Appendix).

Among the previously mentioned six classes, class denoted IV in [4,5] also has interesting properties. Class IV can be represented by the interchange of the two entries $M_0[2,1]$ and $M_0[2,3]$. Generically, the iterations of K *are not integrable*, however for "many" [5] different initial matrices M_0, the orbits of K yield (transcendental [5]) curves. Such a highly regular situation corresponding to (very) weak chaos, has been called "almost integrable" [5]. Moreover there does exist a (codimension-one) algebraic condition, bearing on the entries of the matrix, for which the birational transformations K, associated with class IV, actually correspond to *integrable* mappings [4,5]. This integrability condition has been written elsewhere [4]. The factorizations *restricted to this integrable subcase*[3] read for 4×4 matrices:

$$\begin{aligned} M_{n+1}^{int} &= \frac{K(M_n^{int})}{f_n \; f_{n-2}^2 \; f_{n-3} \; f_{n-4}^2} \\ f_{n+1} &= \frac{det(M_n^{int})}{f_n^2 \; f_{n-1} \; f_{n-2}^3 \; f_{n-3}^2 \; f_{n-4}^3} \end{aligned} \tag{19}$$

where the homogeneous polynomials f_n's verify:

$$\begin{aligned} &\frac{f_{n+2}\, f_{n+7}\, f_{n+9} - f_{n+3}\, f_{n+5}\, f_{n+10}}{f_{n+3}\, f_{n+7}\, f_{n+8} - f_{n+4}\, f_{n+5}\, f_{n+9}} \\ &\qquad = \frac{f_{n+1}\, f_{n+6}\, f_{n+8} - f_{n+2}\, f_{n+4}\, f_{n+9}}{f_{n+2}\, f_{n+6}\, f_{n+7} - f_{n+3}\, f_{n+4}\, f_{n+8}}\cdot \end{aligned} \tag{20}$$

This recursion is an *integrable* recursion on the f_n's. The birational transformations of class IV yield, for this particular integrable case (see (19)), the following (integrable) recursion on the x_n's:

$$\frac{x_{n+2} - 1}{x_{n+1}\, x_{n+3} - 1} = \frac{x_{n+1} - 1}{x_n\, x_{n+2} - 1}\,\frac{x_n\, x_{n+2}}{x_{n+1}}\cdot \tag{21}$$

However, in general, the birational transformations of class IV *only yield*:

$$\frac{x_{n+3} - 1}{x_{n+2}\, x_{n+4} - 1} = \frac{x_{n+1} - 1}{x_n\, x_{n+2} - 1}\, x_n\, x_{n+3}\cdot \tag{22}$$

Recursion (22) is *not* an integrable recursion. Of course (21) implies (22).

Condition (21) cannot really be used as an integrability criterion since it is only verified on a (quite involved codimension-one) algebraic variety [3]. Practically we will use (22) as an integrability criterion: when condition (22) is verified it is easy to find the (codimension-one) algebraic variety on which (21) is actually verified. Thus condition (22) can be seen as an integrability criterion though it is not an integrable recursion. In the following such a situation will be called "class IV-integrability".

Studying the iteration of $\widehat{K}$ in the (q^2-1)-dimensional space $\mathbf{CP}_{q^2-1}$ of $q \times q$ entries, one can show that these orbits actually belong to remarkable two-dimensional subvarieties, namely *planes* [5]:

$$\widehat{K}^{2\,n}(M_0) = a_0 M_0 \; + \; a_1\widehat{K}^2(M_0) \; + \; a_2\widehat{K}^4(M_0)\cdot \tag{23}$$

Inside these planes, (which depend on the initial matrix), the orbits look like curves[4] for many of the trajectories (see [5]). For transposition t_{12-32} one can recursively show [5] that the successive iterates of $\widehat{K}^2$ on a generic matrix M_0, can be written in the following way:

$$\widehat{K}^{2\,n}(M_0) = \frac{1}{x_0\, x_2 \ldots x_{2\,n-2}}\; (M_0 \; + \; \alpha_n \widehat{K}^2(M_0) \; + \; \beta_n P) \tag{24}$$

[3] The factorizations are more involved in the general case [3].

[4] From equations (22) one may have the "prejudice" that the orbits of transformation $\widehat{K}$ in $\mathbf{CP}_{15}$ (or $\mathbf{CP}_{q^2-1}$) should be curves. In fact it has been shown in [5] that, in some domain of the parameter space $\mathbf{CP}_{15}$ (or $\mathbf{CP}_{q^2-1}$), these orbits look like *curves* which may explode into some involved chaotic sets [5].

number of elements	1	2	3	4	5	6	7	8	9	10	11	12
number of classes	2	12	14	34	0	354	0	0	0	0	0	30046

where matrix P denotes the constant $q \times q$ matrix with entries $P[1,2] = 1$, $P[3,2] = -1$, $P[i,j] = 0$ for $(i,j) \neq (1,2)$ or $(i,j) \neq (3,2)$. In other words, all the iterates of $\widehat{K}^2$ *lie in a plane which depends on the initial matrix* M_0 (or equivalently, on any other "even" iterates of M_0). This plane is led by two vectors, namely a *fixed vector* P and another one $\widehat{K}^2(M_0)$, *depending on the initial matrix.*

In the previously mentioned papers [3–5,16,18] all[5] the *integrable* birational mappings have been seen to correspond to the occurrence of one of the three previous hierarchy of recursions on the x_n's represented by (5, 12) or (21) (or (22)). Actually we will, in the following, systematically seek for these three recursions as *integrability criterion*, and verify that the associated hierarchy of recursions [3,4] are also fulfilled. We will verify if one of the three previous factorization schemes (2, 10), or (19), is satisfied, or if some new factorization scheme pops out, *only when* the permutation considered verifies one of the three recursions (5, 12) or (21). This is our "strategy" since the analysis of the factorization schemes cannot be easily implemented as an algorithmic procedure: the factorization schemes are not systematically checked directly for each permutation.

We will also seek for (some) *finite order* recursions on the x_n's and see when these finite order conditions actually correspond to *finite order orbits* of the birational transformation $K = tI$ or, on the contrary, to *infinite order* orbits, and, in this last case, we will carefully examine the orbits: are they elliptic curves, Abelian surfaces, higher dimensional Abelian varieties, chaotic sets... ?

Remark: "Straight" generalizations

It is possible to extend a permutation of $\mathcal{S}_{r\times r}$ to $\mathcal{S}_{q\times q}$ by simply keeping fixed all the new entries corresponding to the additional $q-r$ rows and columns. These permutations on $q\times q$ matrices simply generalizes any permutation introduced on, for instance, a $r \times r$ matrix. Let us write the $q \times q$ matrix in blocks:

$$M_0 = \begin{pmatrix} A_{r,r} & B_{r,q-r} \\ C_{q-r,r} & D_{q-r,q-r} \end{pmatrix}. \qquad (25)$$

Submatrices $A_{r,r}$, $B_{r,q-r}$, $C_{q-r,r}$ and $D_{q-r,q-r}$ are respectively $r \times r$, $r \times (q-r)$, $(q-r) \times r$ and $(q-r) \times (q-r)$ matrices. A simple extension amounts to permuting the entries of the sub-matrix $A_{r,r}$ according to the permutation introduced on the $r \times r$ matrix, and to leaving the others submatrices $B_{r,q-r}$, $C_{q-r,r}$, $D_{q-r,q-r}$ unchanged. We will call such extensions "*straight*" generalizations [16,18].

[5] Except one [3–5].

A certain number of results (occurrence of integrable recursion, factorization schemes...) *do not depend* on the actual size q of the matrix but only on the permutation considered [3,4]. It is thus tempting to examine exhaustively all the permutations of 3×3 (resp. 4×4) matrices in order to *find new integrable birational transformations independent* of q or, on the contrary, find integrable results *specific* of 3×3 (resp. 4×4) matrices. To some extend this last situation could be of a greater interest to get some hint on the very "nature" of integrability.

3 Integrability of an arbitrary permutation of $\mathcal{S}_{3\times 3}$

In this section we extend the previous results to an *arbitrary* permutation t of $\mathcal{S}_{3\times 3}$. Let us first notice that since t is not necessarily an involution anymore, the action of the *group*, generated by t and I, on an initial matrix does *not* reduce[6] to the simple iteration of tI. For instance transformations like t^pI (p integer) do occur. In this section we do not analyse the whole group, generated by I and t: we restrict ourself to the analysis and the classification of the mappings $K = tI$ where t is an arbitrary permutation of $\mathcal{S}_{3\times 3}$. The equivalence relation (9) also holds here. The $9! = 362880$ elements of $\mathcal{S}_{3\times 3}$ are actually grouped in 30462 classes (to be compared to the six classes of the previous section). The number of elements in each class is shown in the table (*see above*).

Notations

Let us introduce some q-independent encoding of the permutations of entries of $q \times q$ matrices. The entries of a 3×3 (resp. $q \times q$) matrix are labeled as follows using hexadecimal representation:

$$\begin{pmatrix} 0 & 3 & 8 \\ 2 & 1 & 7 \\ 6 & 5 & 4 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 3 & 8 & f & \cdots \\ 2 & 1 & 7 & e & \cdots \\ 6 & 5 & 4 & d & \cdots \\ c & b & a & 9 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \qquad (26)$$

A 3×3 (resp. $q\times q$) matrix will represent a permutation of the entries of the 3×3 (resp. $q\times q$) matrix. For instance:

$$P_{erm} = \begin{pmatrix} 2 & 0 & 6 \\ 4 & 7 & 8 \\ 5 & 1 & 3 \end{pmatrix} \qquad (27)$$

[6] Up to a semi-direct product by Z_2. A group generated by two involutions, with no relations between them, is isomorphic to the *infinite dihedral group* [6,7].

Table 1. Permutations corresponding to finite order recursions on the x_n's.

$x_n = \pm 1$	038 (†) 217 [1] 654	026 (†) 315 [1] 874	083 645 [3] 271	062 847 [3] 351	062 351 [6] 847	038 654 [6] 217	172 546 [2] 380	153 748 [2] 260
	153 260 [12] 748	127 564 [12] 308	260 (‡) 153 [4] 748	217 (‡) 654 [4] 038	271 083 [6] 645	206 784 [6] 135	546 380 [2] 172	531 487 [2] 602
$x_n x_{n+1} = \pm 1$	025 316 [6] 874	026 314 [12] 875	024 316 [12] 875	026 814 [12] 375	074 326 [12] 815	062 347 [6] 851	051 347 [12] 862	047 351 [12] 862
	026 351 [6] 847	062 851 [6] 347	153 260 [12] 487	206 134 [12] 785	204 136 [12] 785	260 531 [12] 487	602 (−) 531 [12] 748	
$x_n \cdots x_{n+2}$ $= \pm 1$	047 851 [12] 362	136 204 [12] 785	163 250 [12] 748	150 263 [12] 748	260 758 [12] 143	206 478 [12] 513	607 (−) 531 [12] 482	062 (−) 351 [12] 784
$x_n \cdots x_{n+3}$ $= 1$	067 351 [12] 842	015 374 [12] 826	062 351 [12] 487	$x_n \cdots x_{n+5} = 1$			041 367 [6] 852	186 (‡) 705 [6] 234

corresponds to the permutation which has the following decomposition in a 4-cycle and a 5-cycle.

$$
\begin{aligned}
& 0 \longrightarrow 2 \longrightarrow 4 \longrightarrow 3 \longrightarrow 0 \\
\text{and} \quad & 8 \longrightarrow 6 \longrightarrow 5 \longrightarrow 1 \longrightarrow 7 \longrightarrow 8. \qquad (28)
\end{aligned}
$$

In the following we will also denote by 206 478 513 such a permutation.

For numerical purpose it is more convenient to *study all permutations and check afterward which are equivalent.* This comes from the fact that it is (paradoxically) easier to decide the (5, 12) or (21)-integrability of a given mapping than to decide if two permutations are equivalent under (9).

Remark: equivalence of permutations

An exhaustive list of all the possible permutations of the entries of a 3×3-matrix is given in Table 1 (see above), up to an equivalence relation which is defined as follows. We consider two permutations to be equivalent if they can be related by:

- a relabeling of the variables: permutation of lines and columns i and j,
- a transposition of the matrix[7],
- an inversion: one permutation is equivalent to its inverse[8].

[7] Let us denote tr the transposition of a matrix. One can straightforwardly compare the iteration of PI and $\mathrm{tr}P\,\mathrm{tr}I$.

[8] Analyzing the iteration of $K = PI$ amounts to analyzing the iteration of $K = IP^{-1}$ or equivalently, $K = P^{-1}I$.

At this point, it is useful to make the following remark. Many (quite involved) permutations are seen to satisfy one of the three previous recursions (5, 12) or (21) for "pathological" reasons. Suppose, for instance, that a permutation satisfies the following simple recursion on the x_n's:

$$
x_n x_{n+1} x_{n+2} \; = \; 1. \qquad (29)
$$

It is straightforward to see that (29) implies that recursion (12) is verified. Recursions (5) or (12) or (21) may thus be verified, not because of a "true" integrability (foliation of the parameter space in elliptic, or rational, curves) but because of a simple finite order recursion on the x_n's like (29). Actually it is easy to see that (29) yields $x_n = x_{n+3}$. It can be a strong indication that, not only the recursion on the x_n's, but the birational transformation $\widehat{K}$ *itself*, is a *finite order* mapping. Clearly such finite order recursion on the x_n's has to be considered separately.

As far as such *finite order recursion* on the x_n's are concerned, we have generated *all* the permutations of entries of 3×3 matrices. The permutations (corresponding to *finite order recursions* on the x_n's) are often quite involved permutations. A list of these ("finite order") permutations is given in the following Table 1.

The number in [*bracket*] near each permutation is the cardinality of the corresponding class.The sign (−) indicates that in the first column of Table 1, the minus determination of ± 1 has to be taken. The permutations corresponding to $x_n\, x_{n+1} \cdots x_{n+m-1} = \pm 1$ yield $x_n = x_{n+m}$. The sign (†) refers to (quite trivial permutations) $\widehat{K}^2 =$ identity. The sign (‡) refers to permutations such that $\widehat{K}^6 =$ identity. However most of the permutations of Table 1, corresponding to finite order recursions in the x_n's, *do correspond to infinite order* $\widehat{K}$.

Table 2. "Genuine" class I and class IV recursions.

Class I	028 317 [3,2] 654	046 315 [6,4] 872	027 318 [3,2] 546	027 315 [6,2] 864	025 316 [3,2] 784	016 325 [6,6] 478
	037 218 [3,2] 564	026 347 [3,2] 851	026 345 [3,2] 871	086 345 [6,4] 271	083 647 [3,2] 251	146 308 [6,12] 572
	038 247 [3,2] 651	078 (*) 513 [3,2] 624	147 308 [6,6] 562	062 845 [3,2] 371	206 384 [6,3] 175	047 315 [6,12] 862
	128 307 [3,2] 654	013 625 [6,6] 478	021 465 [6,6] 873	547 380 [6,6] 162	026 (*) 317 [3,2] 854	081 465 [6,6] 273
Class IV	037 218 [6,2] 654	083 245 [6,2] 671	083 645 [12,4] 172	127 465 [12,6] 803		

The factorization schemes of the corresponding birational transformations K of Table 1 often trivialize: only a *finite* number of polynomials f_n's is necessary to describe the factorization scheme (for instance f_1, f_2, f_3). An example of such "trivial" factorization scheme is:

$$M_{2n+1} = \frac{K(M_{2n})}{f_1^n}, \qquad M_{2n+2} = \frac{K(M_{2n+1})}{f_2^n},$$
$$f_{2n+1} = \frac{\det(M_{2n})}{f_1^{2n+1} f_2^n} = 1, \qquad f_{2n} = \frac{\det(M_{2n-1})}{f_1^n f_2^{2n-1}} = 1.$$

The expression of the x_n's, in terms of these finite numbers of f_n's, are also very simple for instance:

$$x_{3p+1} = \frac{f_3}{f_2}, \quad x_{3p+2} = \frac{f_1}{f_3}, \quad x_{3p+3} = \frac{f_2}{f_1},$$
$$\text{yielding:} \quad x_n x_{n+1} x_{n+1} = 1. \qquad (30)$$

If the birational transformation $\widehat{K}$ is a finite order one, one can easily understand that such factorization schemes occur. When the birational transformation $\widehat{K}$ is *not* a finite order transformation it is important to analyse the orbits of the iteration of $\widehat{K}$. We will see in a forthcoming section (Sect. 4) that finite order recursions in the x_n's are not incompatible with *infinite order orbits* of K, and we will provide examples of finite order recursions in the x_n's yielding *elliptic curves* in $\mathbf{CP}_{q^2-1}$.

"Genuine" class I, class IV and class t_1 recursions

Let us now get rid of such "spurious" finite order recursions on the x_n's and concentrate on "true" class t_1, class I and class IV recursions. It is then straightforward to check exactly, for each selected permutation, if the corresponding recursions (5) or (12) or (21) (resp. (22)) hold and, in a second step, if factorizations (2) or (10) or (19) hold.

We have found the following result: *there are no permutation of entries of* 3×3 *matrices verifying* the (sixteen vertex) recursion (5). *This recursion can only be fulfilled for* $q \times q$ matrices for $q \geq 4$.

We have then found 108 permutations verifying recursion (12) (and also a whole hierarchy of recursions [3]) and the factorization scheme (10): all these permutations correspond to integrable mappings for $q = 3$ and can be grouped into 24 classes.

One representative of each of these classes are given in Table 2.

The two numbers in [*bracket*] near each permutation are the cardinality of the corresponding class and the order of the permutation. "Class IV" means the class IV-integrability: the mapping is integrable only for certain initial matrices M_0 of the iteration (codimension-one algebraic variety [3]) and verifies (22).

The representatives in the first column of Table 2 of the these classes are such that their "straight-generalization" to $q \times q$ matrices, also verifies the class I factorization scheme (10) as well as the integrable recursions on the f_n's or on the x_n's for arbitrary value of q (see (11)). Also note that the representatives tagged with a $(*)$ in Table 2 are such that their "straight-q-generalization" also verify, for arbitrary values of q, the class I factorization scheme (10) but not the integrable recursions on the f_n's or on the x_n's: they correspond to a "g-integrability" (*polynomial growth* of the calculations [4]).

One remarks that a large number of integrable mappings *specific* of 3×3 matrices *are not involutions.*

4 Three examples of permutations of 3×3 matrices

4.1 First example: elliptic curves and non involutive permutations of entries of 3×3 matrices associated with finite order recursions $x_n = x_{n+N}$

From Table 1, one sees that there are several possibilities satisfying $x_n x_{n+1} x_{n+2} = 1$ (and thus $x_n = x_{n+3}$), all corresponding to *non-involutive permutations*: for instance three cases correspond to an 8-cycle, two cases correspond to the composition of a 5-cycle and a 2-cycle, and one case corresponds to the composition of a 5-cycle and a 4-cycle.

We focus in this section on the interesting case of the (non involutive) permutation 163250748 of Table 1 satisfying $x_n x_{n+1} x_{n+2} = 1$ and thus $x_n = x_{n+3}$:

$$\mathcal{P} = \begin{pmatrix} 1 & 6 & 3 \\ 2 & 5 & 0 \\ 7 & 4 & 8 \end{pmatrix}. \qquad (31)$$

This permutation is an 8-cycle: $0 \to 1 \to 5 \to 4 \to 8 \to 3 \to 6 \to 7 \to 0$.

Since this permutation corresponds to $x_n = x_{n+3}$, it is not surprising that its factorization scheme trivializes. Actually one can verify that:

$$f_1 = \det(M_0), \quad M_1 = K(M_0), \quad f_2 = \frac{\det(M_1)}{f_1},$$
$$M_2 = K(M_1), \quad f_3 = \frac{\det(M_2)}{f_1^2 f_2}, \quad M_3 = \frac{K(M_2)}{f_1},$$
$$f_4 = \frac{\det(M_3)}{f_1^2 f_2^2 f_3}, \quad M_4 = \frac{K(M_3)}{f_2}, \quad f_5 = \frac{\det(M_4)}{f_1^3 f_2^2 f_3^2},$$
$$M_5 = \frac{K(M_4)}{f_1 f_3}, \quad f_6 = 1 = \frac{\det(M_5)}{f_1^3\, f_2^3\, f_3^2}. \qquad (32)$$

One also has the following result for arbitrary integer n:

$$\widehat{K}^{n+18}(M_0) - \widehat{K}^n(M_0) = \lambda \times (\widehat{K}^{n+12}(M_0) - \widehat{K}^{n+6}(M_0)) = \Delta(M_0, n) \qquad (33)$$

where the two first rows of Δ are equal. From (32) and (33) one could expect that the orbit of the birational transformation $\widehat{K}$ are trivial. In fact the iteration of $\widehat{K}$ yields *elliptic curves* as can be seen in Figure 1.

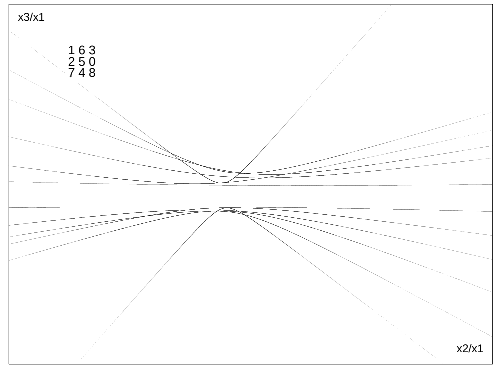


Fig. 1. Projection on two variables of $\mathbf{CP}_8$ of an orbit of $\widehat{K}^2$ for permutation 163 250 748 (finite order recursion).

4.2 Second example: elliptic curves and non involutive permutations of class I

Let us now consider the permutations of class I (see Tab. 1). Most of them are *non-involutive*. Remarkably, most of the permutations of class I give *elliptic curves* (and not higher dimensional Abelian varieties, ...). As an example, let us consider the non involutive permutation 086 345 271 of Table 1:

$$\mathcal{P} = \begin{pmatrix} 0 & 8 & 6 \\ 3 & 4 & 5 \\ 2 & 7 & 1 \end{pmatrix}. \qquad (34)$$

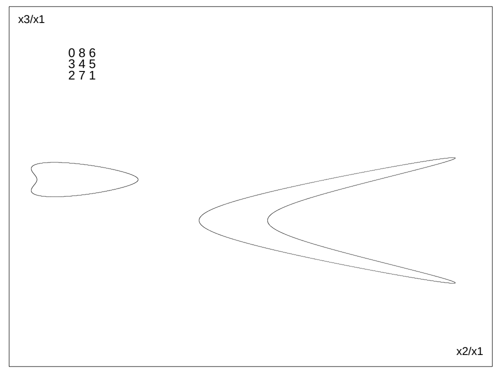


Fig. 2. Projection on two variables of $\mathbf{CP}_8$ of an orbit of $\widehat{K}^2$ for permutation 086 345 271 (class I recursion non involutive).

It can be seen to decompose into a product of a 4-cycle and two 2-cycles: $0 \to 3 \to 8 \to 6 \to 2 \to 3$, $1 \to 4 \to 1$ and $5 \to 7 \to 5$. It corresponds to class I and is *specific* of $q = 3$ (namely the "straight generalization" of this permutation does not yield recursion of class I).

One has the following factorization scheme for this non-involutive permutation:

$$f_1 = \det(M_0), \quad M_1 = K(M_0), \quad f_2 = \det(M_1),$$
$$M_2 = K(M_1), \quad f_3 = \det(M_2), \quad M_3 = K(M_2),$$
$$f_4 = \frac{\det(M_2)}{f_1^2}, \quad M_4 = \frac{K(M_3)}{f_1}, \quad f_5 = \frac{\det(M_3)}{f_2^2},$$
$$M_5 = \frac{K(M_4)}{f_2}, \cdots \qquad (35)$$

and for arbitrary n:

$$M_{n+3} = \frac{K(M_{n+2})}{f_n}, \quad f_{n+3} = \frac{\det(M_{n+2})}{f_n^2},$$
$$\widehat{K}(M_{n+2}) = \frac{K(M_{n+2})}{\det(M_{n+2})} = \frac{M_{n+3}}{f_n f_{n+3}}. \qquad (36)$$

The f_n's *actually satisfy the integrable recursion* (11):

$$\frac{f_n f_{n+3}^2 - f_{n+4} f_{n+1}^2}{f_{n-1} f_{n+3} f_{n+4} - f_n f_{n+1} f_{n+5}} = \frac{f_{n-1} f_{n+2}^2 - f_{n+3} f_n^2}{f_{n-2} f_{n+2} f_{n+3} - f_{n-1} f_n f_{n+4}} \qquad (37)$$

and the integrable recursion (12) on the x_n's is also verified. Studying the iteration of $\widehat{K}^2$ in the (3^2-1)-dimensional projective space $\mathbf{CP}_{3^2-1} = \mathbf{CP}_8$, one can show that these orbits correspond to *elliptic curves*: this is shown on Figure 2. Furthermore these elliptic curves belong to a three-dimensional vectorspace (four homogeneous variables): relation (15) is still valid here. Similarly to the calculations detailed in the appendix for transposition t_{12-21}, one can thus find an infinite number of representation of $\widehat{K}^2$ as a birational mapping of three variables (four homogeneous variables). One can also see that the following constant matrix P:

$$P = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (38)$$

(such that $\mathcal{P}^{\in}(\mathcal{P}) = -\mathcal{P}$) always belongs to this three-dimensional vectorspace. Thus $\widehat{K}^6(M_0)$, in the right-hand side of relation (15), can again be replaced by P. Therefore the orbits of $\widehat{K}^2$ belong to the following three-dimensional vectorspace:

$$\widehat{K}^{2\,n}(M_0) = a_0 M_0 + a_1 \widehat{K}^2(M_0) + a_2 \widehat{K}^4(M_0) + a_3 P. \qquad (39)$$

This property[9] is exactly the same (with a slight modification of the base point) as the one noticed for permutation t_{12-21} (see (15) and (A.1)). This situation has to be compared with the one of the permutations associated with *finite order recursions* of the previous sections (see relations (33)).

Since we have a foliation in *elliptic curves* of the *whole* eight dimensional parameter space $\mathbf{CP}_8$ of the entries of the 3×3 matrices, the point corresponding to matrix P can be seen to be similar to a *base points* of an elliptic foliation [7]. Each elliptic curve (in $\mathbf{CP}_{q^2-1}$), generated by the iteration of $\widehat{K}$, can be seen to "pop out" from P.

One can actually use a relation such as (39) in order to get exhaustively all the constant matrices (base points of the elliptic foliation) belonging to all the elliptic curves of the elliptic foliation. Actually if one considers several initial matrices $M_0^{(1)}, M_0^{(2)}, \ldots, M_0^{(r)}, \ldots$ a constant matrix like matrix P should belong to the various eigenspaces associated to these various initial matrices $M_0^{(m)}$. Therefore there should exist coefficients $a_m^{(n)}$ such that:

$$P = a_0^{(1)} M_0^{(1)} + a_1^{(1)} \widehat{K}^2(M_0^{(1)}) + a_2^{(1)} \widehat{K}^4(M_0^{(1)}) + a_3^{(1)} \widehat{K}^6(M_0^{(1)}) = \cdots \qquad (40)$$

$$= a_0^{(r)} M_0^{(r)} + a_1^{(r)} \widehat{K}^2(M_0^{(r)}) + a_2^{(r)} \widehat{K}^4(M_0^{(r)}) + a_3^{(r)} \widehat{K}^6(M_0^{(r)}) = \cdots$$

This linear system can easily be solved to prove that matrix P is, for permutation (34), the *only* such "base point" matrix with finite entries.

Remark

The occurrence of three-dimensional vectorspaces like (15), can, in fact, be seen to be valid for most (see below) of the class I permutations of Table 2, these permutations being, or not, involutive. Let us however note that there exist some exceptions like the permutations (of the classes) corresponding to the last column of Table 2, for which the occurrence of three-dimensional vectorspaces like (15) is ruled out. Their $\widehat{K}$-orbits actually live in a *six-dimensional* vectorspace:

$$\widehat{K}^{2\,n}(M_0) = a_0 M_0 + a_1 \widehat{K}^2(M_0) + a_2 \widehat{K}^4(M_0) + a_3 \widehat{K}^6(M_0) + a_4 \widehat{K}^8(M_0) + a_5 \widehat{K}^{10}(M_0) + a_6 \widehat{K}^{12}(M_0). \qquad (41)$$

Introducing the inverse of transformation $\widehat{K}$, namely $\widehat{L} = \widehat{I}\mathcal{P}^{-1} = \widehat{I}\mathcal{P}^5$, relation (41) can be written in a more "balanced" way:

$$\widehat{K}^{2\,n}(M_0) = b_3 \widehat{L}^6(M_0) + b_2 \widehat{L}^4(M_0) + b_1 \widehat{L}^2(M_0) + a_0 M_0 + a_1 \widehat{K}^2(M_0) + a_2 \widehat{K}^4(M_0) + a_3 \widehat{K}^6(M_0). \qquad (42)$$

We have not been able to find a "base point" for these permutations.

Again, for all these last permutations, property (41) is not restricted to the points of the orbit of $\widehat{K}^2$, namely the $\widehat{K}^{2\,n}(M_0)$'s. The *six-dimensional* vectorspace $\mathcal{V}_6$ corresponding to the right-hand side of (41) is *actually invariant* by $\widehat{K}^2$: $\widehat{K}^2(\mathcal{V}_6) = \mathcal{V}_6$. Therefore *one has a representation of* $\widehat{K}^2$ as a (birational integrable) mapping on *six* variables or, seen as a homogeneous polynomial transformation, of seven homogeneous parameters. These *six-dimensional* representations are too involved to be given here for a generic initial matrix M_0. The calculations for a *given* initial matrix M_0 are similar to the ones detailed in Appendix. Again one obtains an *infinite set* of (birational integrable) mapping on *six* variables (as many as the number of initial matrices M_0 one wants to consider). For instance, for the permutations of the last column of Table 2, one obtains integrable mappings of *seven* homogeneous variables $a_0, \ldots, a_6$ represented in terms of *quartic homogeneous polynomial transformations*:

$$a_i \quad \longrightarrow \quad \sum_\alpha A_\alpha^{(i)} a_0^{\alpha_0} a_1^{\alpha_1} a_2^{\alpha_2} a_3^{\alpha_3} a_4^{\alpha_4} a_5^{\alpha_5} a_6^{\alpha_6},$$

[9] For $q = 4$ one can verify that relation (15), or (39), does not hold anymore for the "straight generalization" of permutation 086 345 271.

$$i = 0, 1, 2, \ldots, 6 \tag{43}$$

where α denotes sets of seven exponents $\alpha_0, \alpha_1, \cdots, \alpha_6$ such that $\alpha_0 + \alpha_1 + \cdots + \alpha_6 = 4$.

4.3 Third example: non involutive permutations of class IV

There actually exists *non-involutive permutations* of Table 2, yielding recursion (22):

$$\begin{pmatrix} 0\,8\,3 \\ 6\,4\,5 \\ 1\,7\,2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1\,2\,7 \\ 4\,6\,5 \\ 8\,0\,3 \end{pmatrix} \cdot \tag{44}$$

The first permutation is the product of a 4-cycle and of two involutions. The second one is the product of a 6-cycle and of a 3-cycle.

Let us recall relation (23) for transposition $M[2,1] \leftrightarrow M[2,3]$ of class IV. For this last transposition $M[2,1] \leftrightarrow M[2,3]$ the orbits of $\widehat{K}^{2n}$ lie in planes (see (23) and [5]) depending non trivially of the initial matrix M_0. This property *is also valid for the two non involutive permutations* (44) (but here for $q = 3$ only). Note that, for these two *non involutive* permutations (44), there also exist a simple "base point" P such that (24) namely:

$$P = \begin{bmatrix} 0 & 0\,0 \\ 0 & 0\,0 \\ -1 & 0\,1 \end{bmatrix} \quad \text{for} \quad \begin{pmatrix} 0\,8\,3 \\ 6\,4\,5 \\ 1\,7\,2 \end{pmatrix}$$

$$\text{and} \qquad P = \begin{bmatrix} 1\,0 & -1 \\ 0\,0 & 0 \\ 0\,0 & 0 \end{bmatrix} \quad \text{for} \quad \begin{pmatrix} 1\,2\,7 \\ 4\,6\,5 \\ 8\,0\,3 \end{pmatrix} \cdot$$

One can verify[10] that these matrices P are *the only constant matrices* such that (24) is satisfied for arbitrary initial matrix M_0 (see relation (40)). In contrast with transposition $M[2,1] \leftrightarrow M[2,3]$ of class IV, the visualization of the orbits is disappointing for these two non-involutive permutations.

5 Integrability of permutations of $\mathcal{S}_{4\times 4}$

Similar calculations can be performed for 4×4 matrices. It is for instance interesting to find integrable mappings *specific* of 4×4 matrices or, on the contrary, which can be generalized to $q \times q$ matrices (with $q \geq 5$ but different from these upgraded 3×3 involutive examples). This justifies performing the previous exhaustive analysis but for arbitrary permutations of entries of 4×4 matrices.

Since the number of permutations of $\mathcal{S}_{4\times 4}$ is very large to explore (16! = 20 922 789 888 000) we have first investigated only 16!/4! = 871 782 912 000 permutations corresponding to an *ordering constraint on the diagonal entries.* With the previous encoding (26) of the permutations the diagonal symbols (namely $0, 1, 4, 9$) become, after transformation by a permutation $\mathcal{P}$: $\mathcal{P}(0), \mathcal{P}(1), \mathcal{P}(4), \mathcal{P}(9)$. The constraint on the diagonal entries is the following: $\mathcal{P}(0) \leq \mathcal{P}(1) \leq \mathcal{P}(4) \leq \mathcal{P}(9)$. This constraint divides by 24 the computer time[11]. Using a twelve processors dedicated machine running for six months we have scanned the 871 782 912 000 permutations. Performing tests in standard precision we have discarded most of the previous permutations to select, in a first step, around eight thousand permutations likely to be class I or class IV or class t_1. Eventually using an infinite precision rational representation we have determined, among the remaining permutations, those exactly belonging to class I or class IV or class t_1.

When a new permutation is found to verify one of the three previous recursions (5, 12) or (22), one checks if this permutation is, or is not, equivalent (up to relabeling (9)) to another one already found. For instance this reduces the number of permutations found for "class I-integrability" (see Tab. 3) to only 348 non-equivalent permutations (classes). However it should be noticed that some classes *do occur simultaneously* in class I and class t_1 (and possibly class IV). They, in fact, satisfy simple finite order recursions on the x_n's (like $x_n x_{n+1} = 1, \ldots$). More precisely 199 classes occur simultaneously in class I and class t_1 *and also in* class IV, 81 classes occur simultaneously in class I and class t_1 *but not in* class IV. The results are summarized in Table 3.

Table 3. New number of permutations.

	Class I	Class IV	Class t_1
number of classes	348	205	327
genuine classes	68	6	47

Integrability can be seen to be a quite "rare" phenomenon: collecting the genuine class I, class IV and class t_1 integrability, as well as the classes corresponding to finite order recursion on the x_n's, integrability corresponds to less than 2×10^4 permutations among 16! permutations, that is a ratio of $\simeq 10^{-9}$. This ratio has to be compared with the ratio emerging from the exhaustive analysis of 3×3 matrices: 27 classes among 30 462 classes that is a ratio of $\simeq 0.886 \times 10^{-4}$.

[10] For $q = 4$ one easily verify that relation (23) or (24) does not hold anymore for the "straight generalization" of permutation 083 645 172 as well as for permutation 127 465 803. This is not surprising since we got in Table 2 that permutation 083 645 172 satisfies the class IV recursions *specifically* for $q = 3$.

[11] It restricts the number of "integrable" permutations (almost by 24) but does not restrict drastically the number of classes since each class is likely to have a representative which satisfies this constraint (this can be easily seen on the previous example of permutations of entries of 3×3 matrices).

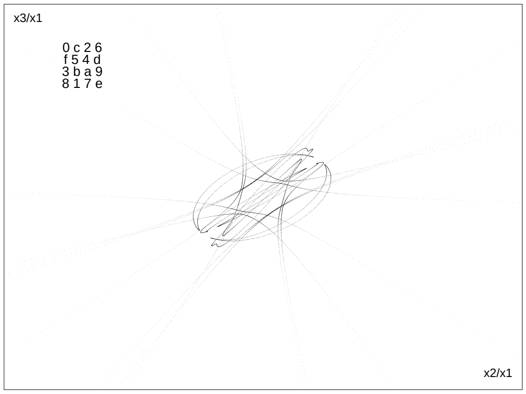


Fig. 3. Projection on two variables of an orbit of $\widehat{K}^2$ for permutation $0c26\,f54d\,3ba9\,817e$ (class I recursion for 4×4 matrices).

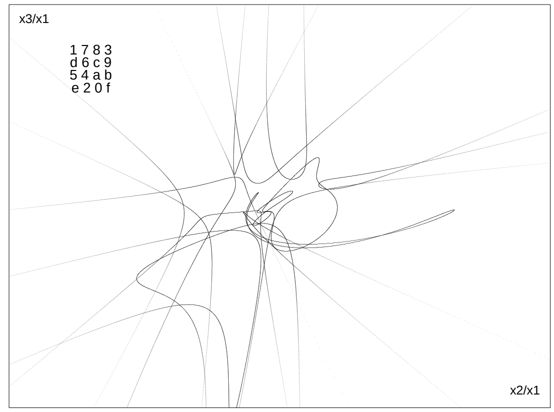


Fig. 4. Projection on two variables of an orbit of $\widehat{K}^2$ for permutation $1783\,d6c9\,54ab\,e20f$ (sixteen vertex recursion for 4×4 matrices).

The exhaustive list of permutations of Table 3 is available by ftp [21].

Systematic visualization of this exhaustive list

We have systematically visualized the orbits corresponding to this exhaustive list of classes of permutations. We have obtained the following *preliminary* results.

As far as this visualization approach is concerned, the permutations of class IV give poor results. This is not surprising since class IV is not generically integrable but only on a *codimension-one* algebraic variety [5]. So let us concentrate, in a first step, on the permutations of class I and class t_1 and then to some permutations corresponding to "some" *finite order recursions* on the x_n's (however in this last set of permutations we do not try, like for class I, class IV and class t_1, to get them exhaustively: we just give examples).

Class I

Let us first consider the permutations genuinely of class I.

Among these 68 classes a set of (at least) sixteen classes of permutations clearly correspond to *elliptic curves* (see Fig. 3). We give below an (arbitrary) representative of each class:

$$\begin{pmatrix} 0 & c & 2 & 6 \\ f & 5 & 4 & d \\ 3 & b & a & 9 \\ 8 & 1 & 7 & e \end{pmatrix} \begin{pmatrix} 0 & 2 & 1 & 7 \\ 8 & 6 & 5 & 4 \\ 3 & c & b & a \\ f & e & 9 & d \end{pmatrix} \begin{pmatrix} 0 & 2 & 3 & 8 \\ 6 & 7 & 5 & 4 \\ c & 1 & b & a \\ f & e & 9 & d \end{pmatrix} \begin{pmatrix} 0 & f & 3 & 8 \\ 6 & 7 & a & 4 \\ c & 1 & b & 5 \\ 2 & e & 9 & d \end{pmatrix}$$
$$\begin{pmatrix} 0 & 2 & c & 6 \\ 8 & 7 & 5 & 4 \\ 3 & 1 & b & a \\ f & e & 9 & d \end{pmatrix} \begin{pmatrix} 0 & f & c & 6 \\ 8 & 7 & a & 4 \\ 3 & 1 & b & 5 \\ 2 & e & 9 & d \end{pmatrix} \begin{pmatrix} 0 & f & 8 & 3 \\ c & 7 & a & 4 \\ 6 & 1 & b & 5 \\ 2 & e & 9 & d \end{pmatrix} \begin{pmatrix} 0 & 6 & c & 2 \\ f & 7 & 1 & e \\ 3 & a & b & 9 \\ 8 & 4 & 5 & d \end{pmatrix}$$
$$\begin{pmatrix} 2 & c & 6 & 0 \\ 8 & 3 & d & f \\ a & b & 5 & 9 \\ 7 & 1 & 4 & e \end{pmatrix} \begin{pmatrix} 2 & c & 6 & 0 \\ b & a & 4 & 9 \\ 3 & 8 & d & f \\ 1 & 7 & 5 & e \end{pmatrix} \begin{pmatrix} 3 & 7 & 2 & 1 \\ 8 & 4 & 6 & 5 \\ 0 & a & c & b \\ f & d & 9 & e \end{pmatrix} \begin{pmatrix} 3 & 5 & 1 & b \\ 0 & 6 & 2 & c \\ 9 & 4 & 7 & f \\ a & d & e & 8 \end{pmatrix}$$
$$\begin{pmatrix} 3 & 5 & 1 & b \\ 0 & 6 & 2 & c \\ 9 & 4 & 7 & a \\ f & d & e & 8 \end{pmatrix} \begin{pmatrix} 3 & 4 & 0 & d \\ 5 & 8 & 6 & f \\ c & a & b & 9 \\ 2 & 7 & 1 & e \end{pmatrix} \begin{pmatrix} 4 & 1 & 2 & d \\ 7 & 5 & 6 & e \\ a & c & b & 9 \\ 8 & 0 & 3 & f \end{pmatrix} \begin{pmatrix} 5 & 2 & 6 & 4 \\ 3 & 7 & 0 & 8 \\ b & 1 & c & a \\ 9 & e & d & f \end{pmatrix}.$$

One verifies for these last permutations the *same* factorization scheme as class I, namely (10), and the same recursions on the x_n's, or f_n's or q_n's (see (11), (12), (13)).

Though most of these permutations are *not involutive* (and thus should yield drastically more involved results) the results corresponding to the associated birational transformation K cannot be distinguished from the one of transposition t_{12-21}. Fifteen other classes clearly seem to also correspond to foliations in elliptic curves. The remaining permutations yield more disappointing orbits (like the ones corresponding to the last column of Tab. 2) and should be revisited by other (exact) methods.

Class t_1

Among the 47 permutations of class t_1, the six permutations given below have the same factorization scheme (see (2)) and the same hierarchy of recursion of the f_n's (3):

$$\begin{pmatrix} 0 & 4 & 6 & 8 \\ 9 & 1 & e & b \\ c & 7 & 2 & a \\ f & 5 & d & 3 \end{pmatrix} \begin{pmatrix} 0 & 2 & 7 & 8 \\ 3 & 1 & e & f \\ c & 6 & 4 & a \\ b & 5 & d & 9 \end{pmatrix} \begin{pmatrix} 0 & c & a & 8 \\ 2 & 6 & 4 & 7 \\ 3 & b & 9 & f \\ 1 & 5 & d & e \end{pmatrix}$$
$$\begin{pmatrix} 0 & a & c & 8 \\ 6 & 7 & 2 & 4 \\ 3 & 9 & b & f \\ 5 & e & 1 & d \end{pmatrix} \begin{pmatrix} 1 & 3 & 2 & 0 \\ b & 5 & c & 6 \\ a & 4 & 9 & d \\ 7 & 8 & e & f \end{pmatrix} \begin{pmatrix} 1 & 7 & 8 & 3 \\ d & 6 & c & 9 \\ 5 & 4 & a & b \\ e & 2 & 0 & f \end{pmatrix}.$$

Typical orbits for any permutation of these six classes, as well as for permutations of twenty-two other classes not given here, look like the orbits presented on Figure 4. They are elliptic curves. The nineteen remaining t_1-classes also deserve further analysis.

Let us recall that $199 + 81$ classes among the global set of 327 and 348 classes of respectively class-t_1 and class I, correspond to finite order recursions on the x_n's. They are discussed below.

Finite order recursion on the x_n's

Some permutations correspond to trivialization of the factorization scheme (see (32)), to a *periodicity* in the x_n's and clearly yield curves. For instance the following permutations yield $x_n = x_{n+3}$:

$$\begin{pmatrix} 0&4&c&7\\ 3&5&b&1\\ 8&6&a&2\\ f&d&9&e \end{pmatrix} \begin{pmatrix} 0&6&a&7\\ 3&5&b&1\\ 8&4&c&2\\ f&d&9&e \end{pmatrix} \begin{pmatrix} 0&2&c&6\\ 8&7&a&4\\ 3&1&b&5\\ f&9&d&e \end{pmatrix} \begin{pmatrix} 1&5&3&b\\ 2&6&0&c\\ 7&d&8&9\\ e&4&f&a \end{pmatrix}$$

$$\begin{pmatrix} 1&b&4&3\\ 7&a&5&8\\ 2&c&d&0\\ e&9&6&f \end{pmatrix} \begin{pmatrix} 2&6&0&c\\ 1&5&3&b\\ 7&4&8&a\\ f&9&e&d \end{pmatrix} \begin{pmatrix} 3&5&b&1\\ 8&4&a&7\\ 0&6&c&2\\ 9&e&f&d \end{pmatrix} \begin{pmatrix} 3&5&b&1\\ 0&6&c&2\\ f&d&a&7\\ 8&4&9&e \end{pmatrix}$$

and the following ones correspond to $x_n = x_{n+2}$:

$$\begin{pmatrix} 1&3&b&5\\ 0&2&6&c\\ e&f&9&d\\ 8&7&4&a \end{pmatrix} \begin{pmatrix} 1&6&b&0\\ 2&5&c&3\\ e&4&9&8\\ 7&d&a&f \end{pmatrix} \begin{pmatrix} 2&0&6&c\\ 1&3&d&9\\ 7&8&5&b\\ e&f&4&a \end{pmatrix} \begin{pmatrix} 2&5&c&3\\ 1&6&b&0\\ e&4&9&8\\ 7&d&a&f \end{pmatrix}$$

$$\begin{pmatrix} 2&f&9&6\\ 1&8&a&5\\ 7&3&b&4\\ e&0&c&d \end{pmatrix} \begin{pmatrix} 6&f&9&2\\ 5&8&a&1\\ 4&3&b&7\\ d&0&c&e \end{pmatrix} \begin{pmatrix} 0&2&6&c\\ 3&1&d&9\\ 8&7&5&b\\ f&e&4&a \end{pmatrix}.$$

The permutations corresponding to $x_n = x_{n+3}$ clearly yield $\widehat{K}^2$-orbits which are *elliptic curves* but the ones corresponding to $x_n = x_{n+2}$ could even yield rational curves. Some permutations seem to give orbits which "look like surfaces". They correspond to $x_n = x_{n+3}$:

$$\begin{pmatrix} 1&6&0&b\\ 2&5&3&c\\ e&4&8&9\\ 7&d&f&a \end{pmatrix} \begin{pmatrix} 1&3&b&5\\ 4&a&8&7\\ 2&0&c&6\\ d&9&f&e \end{pmatrix} \begin{pmatrix} 2&c&6&0\\ 5&3&1&b\\ 4&8&7&a\\ e&9&d&f \end{pmatrix} \begin{pmatrix} 3&5&c&2\\ 0&6&b&1\\ f&d&a&7\\ 8&4&9&e \end{pmatrix}.$$

Some seem to give orbits which look like surfaces or volumes. They correspond respectively to $x_n = x_{n+3}$:

$$\begin{pmatrix} 1&3&b&5\\ e&8&9&4\\ 7&0&a&6\\ 2&f&c&d \end{pmatrix} \begin{pmatrix} 2&3&c&5\\ e&8&9&4\\ 1&0&b&6\\ 7&f&a&d \end{pmatrix} \begin{pmatrix} 7&3&a&5\\ 2&8&c&4\\ 1&0&b&6\\ e&f&9&d \end{pmatrix} \begin{pmatrix} 1&0&b&6\\ 7&3&a&5\\ 2&8&c&4\\ e&f&9&d \end{pmatrix}$$

and $x_n = x_{n+2}$:

$$\begin{pmatrix} 2&5&c&3\\ 7&6&a&0\\ 1&4&b&8\\ e&d&9&f \end{pmatrix} \begin{pmatrix} 5&0&c&1\\ 6&8&a&2\\ 4&3&b&7\\ d&f&9&e \end{pmatrix}.$$

Some deserve further studies (most of the time they probably yield elliptic curves, maybe surfaces...). Among this set one has the following permutations yielding $x_n = x_{n+2}$:

$$\begin{pmatrix} 0&5&1&c\\ 3&6&2&b\\ f&4&7&9\\ 8&d&e&a \end{pmatrix} \begin{pmatrix} 0&2&c&6\\ 8&7&a&4\\ 3&1&b&5\\ e&f&9&d \end{pmatrix} \begin{pmatrix} 1&4&b&8\\ 2&5&c&3\\ 7&6&a&0\\ e&d&9&f \end{pmatrix} \begin{pmatrix} 1&5&b&3\\ 7&4&a&8\\ 2&6&c&0\\ 9&f&e&d \end{pmatrix}$$

$$\begin{pmatrix} 1&5&b&3\\ 2&6&c&0\\ a&8&7&4\\ e&d&9&f \end{pmatrix} \begin{pmatrix} 1&5&b&3\\ 2&6&c&0\\ 7&4&a&8\\ 9&f&e&d \end{pmatrix} \begin{pmatrix} 2&0&c&6\\ 1&3&b&5\\ a&4&7&8\\ 9&d&e&f \end{pmatrix} \begin{pmatrix} 2&6&c&0\\ b&3&1&5\\ a&8&7&4\\ e&d&9&f \end{pmatrix}$$

$$\begin{pmatrix} 2&0&c&6\\ 1&3&b&5\\ 8&7&a&4\\ f&e&9&d \end{pmatrix} \begin{pmatrix} 2&0&c&6\\ 1&3&b&5\\ 8&7&a&4\\ e&f&9&d \end{pmatrix} \begin{pmatrix} 2&c&0&6\\ 5&3&b&1\\ 4&8&a&7\\ e&9&f&d \end{pmatrix} \begin{pmatrix} 2&f&9&6\\ 7&3&b&4\\ 1&8&a&5\\ e&0&c&d \end{pmatrix}$$

$$\begin{pmatrix} 2&6&0&b\\ 1&5&3&c\\ 7&4&8&9\\ e&d&f&a \end{pmatrix} \begin{pmatrix} 2&0&c&6\\ b&5&1&3\\ 7&8&a&4\\ 9&d&e&f \end{pmatrix} \begin{pmatrix} 2&c&0&6\\ 4&8&a&7\\ 5&3&b&1\\ e&9&f&d \end{pmatrix} \begin{pmatrix} 2&0&c&6\\ 7&8&a&4\\ 1&3&b&5\\ d&9&f&e \end{pmatrix}$$

$$\begin{pmatrix} 2&0&c&6\\ 7&8&a&4\\ 1&3&b&5\\ 9&d&e&f \end{pmatrix} \begin{pmatrix} 6&c&0&2\\ 7&8&a&4\\ 1&3&b&5\\ d&9&f&e \end{pmatrix} \begin{pmatrix} 0&2&6&b\\ 3&1&5&c\\ 8&7&4&9\\ f&e&d&a \end{pmatrix} \begin{pmatrix} 1&5&b&3\\ 2&6&c&0\\ a&8&7&4\\ 9&f&e&d \end{pmatrix}$$

$$\begin{pmatrix} 0&2&c&6\\ 1&3&b&5\\ 8&7&a&4\\ e&f&9&d \end{pmatrix} \begin{pmatrix} 1&3&5&b\\ 0&2&c&6\\ f&e&9&d\\ 7&8&4&a \end{pmatrix} \begin{pmatrix} 1&0&c&5\\ 2&3&b&6\\ 7&8&a&4\\ e&f&9&d \end{pmatrix}$$

and permutations yielding $x_n = x_{n+3}$:

$$\begin{pmatrix} 0&c&6&2\\ 8&7&a&4\\ 3&1&b&5\\ f&e&9&d \end{pmatrix} \begin{pmatrix} 1&5&b&3\\ 2&6&c&0\\ a&8&7&4\\ e&d&9&f \end{pmatrix} \begin{pmatrix} 6&0&c&2\\ 4&8&a&7\\ 5&3&b&1\\ e&d&9&f \end{pmatrix} \begin{pmatrix} 0&1&b&5\\ 8&2&c&6\\ 3&7&a&4\\ f&e&9&d \end{pmatrix}.$$

Clearly this first visualization approach needs further analysis in order to know, in a precise way, the actual *dimension* of the orbits (surface, volume, higher dimensional Abelian varieties...). In particular one certainly needs to obtain the equations of these orbits introducing the some Plücker-like variables [13]. This is in progress.

6 Conclusion

The remarkable integrability properties of permutation t_1, which corresponds to the sixteen vertex model, and transposition t_{12-21} (the two associated integrable mappings being in fact closely related [3,4]) have been seen to be shared by a surprisingly large set of permutations of entries of 3×3 (for t_{12-21}) and 4×4 (for t_1) matrices, even *non-involutive* ones. A systematic analysis of these permutations led us to discover a quite large set of permutations that could present some interest for lattice models, in particular *vertex models* in d dimension, d being *not necessarily two*. The associated birational mappings are, of course, interesting *per se*, as discrete dynamical systems of many variables. As far as lattice statistical mechanics is concerned, we are in the amusing situation where we have "the result before having the problem", or, more precisely, we have the parametrization of the vertex model before knowing the R-matrix of the vertex model: a permutation $\mathcal{P}$ of a 4×4 matrix can represent a geometrical (reflection, rotation ...) symmetry of a sixteen vertex model R-matrix, or a geometrical symmetry of a vertex model on a cubic lattice where the $2^3 \times 2^3$ R-matrix can be written as two 4×4 identical blocks [16], or could represent the "reduction" of a hyperplane reflection symmetry t_1 on a d-dimensional hypercubic lattice in the case where the $q^d \times q^d$ R-matrix can be reduced (after relabeling) to the direct sum of 4×4 matrices.

If two permutations t and t' yield integrable mappings K_t and $K_{t'}$, their product $t'' = t \cdot t'$ does not yield (in general) an integrable mappings $K_{t''}$ (or even a "g-integrable" mapping). The set of "integrable" permutations does not have any obvious group structure, or even algebraic, or graph, or combinatorial structure. The "universality classes" of integrable birational mappings (or g-integrable ones) seem, after the systematic analysis sketched in this paper, larger and also more "subtle", that one could expect. In particular the non-involutive character of the permutation does not seem to be such a crucial prerequisite for integrability. The non-involutive examples we have obtained, should be precious to understand the nature of integrability since *we do not have a (birational) representation of the infinite dihedral group*[12] *anymore* [6]. The occurrence, for integrable mappings, of *non involutive* permutations could mean that the integrability of discrete dynamical systems is a "larger" concept than the integrability of the symmetries of lattice (d-dimensional) models. Is it possible to find a new "point of view" such that we could make a distinction between a truly non-involutive character of a permutation and non-involutive characters that are in fact irrelevant, thus defining the "universality classes" of integrable models?

We have not encountered any example of permutation satisfying an (integrable) class I or class t_1-recursion on the auxiliary variables x_n's (or f_n's...) yielding orbits for $\widehat{K}$ which are algebraic varieties of dimension greater than one (Abelian surfaces...). Apparently all the higher dimensional Abelian algebraic varieties of $\widehat{K}$ correspond to situations where one encounters a periodicity in the x_n's: $x_n = x_{n+N}$. This fully justifies a systematic analysis of these x_n-periodic situations which also yield elliptic curves for the iteration of $\widehat{K}$. The integrability related to *finite order* recursions on the x_n's, namely $x_n = x_{n+N}$, seems to yield a "whole universe" of new g-integrable birational mappings $\widehat{K}$. It would be interesting to clearly identify the dimension of these higher dimensional algebraic varieties and understand the parameters this dimension depends on. In this respect let us recall that a *polynomial growth* of the calculation can be associated to elliptic curves or Abelian surfaces (Abelian algebraic varieties...) as well!

We hope that the accumulation of results from such very large computer calculations will provide many more examples of *non-involutive "integrable"* permutations which should help to better understand the structures associated with integrability.

[12] The Coxeter groups generated by these non-involutive permutations and the matrix inversion (namely I) certainly deserve to be analysed [20].

Appendix: infinite number of quartic integrable mappings associated with transposition t_{12-21}

Let us recall for transposition t_{12-21} relation (A.1) on the homogeneous transformation K^2:

$$K^{2\,n}(M_0) = a_0 M_0 + a_1 K^2(M_0) + a_2 K^4(M_0) + a_3 P. \tag{A.1}$$

Let us see that this relation is not restricted to points of the orbit of M_0 under the iteration of K^2, like $K^{2\,n}(M_0)$, but that it does correspond to a stability of the three-dimensional vectorspace $\mathcal{V}_3$ spanned by M_0, $K^2(M_0)$, $K^4(M_0)$ and the constant matrix P. Actually let us consider an arbitrary linear combination of M_0, $K^2(M_0)$, $K^4(M_0)$ and the constant matrix P which is not of the form $K^{2\,n}(M_0)$ for some integer n:

$$M_{\mathcal{V}_3} = a_0 M_0 + a_1 K^2(M_0) + a_2 K^4(M_0) + a_3 P. \tag{A.2}$$

One can show that the image, under transformation K^2, of this generic point of $\mathcal{V}_3$ *also belongs to* $\mathcal{V}_3$:

$$K^2(M_{\mathcal{V}_3}) = A_0 M_0 + A_1 K^2(M_0) + A_2 K^4(M_0) + A_3 P \tag{A.3}$$

where the new A_i's are *quartic* homogeneous expressions of the previous a_i's, the coefficients of these *quartic* expressions depending, in a quite involved way, of the entries of the initial matrix M_0. The expression of these coefficients are too involved, and too huge, to be given for a generic initial matrix M_0. Thus, let us just consider a particular initial matrix, for instance:

$$M_0 = \begin{bmatrix} 3 & 0 & 2 \\ -1 & 2 & 7 \\ 1 & 1 & 2 \end{bmatrix}. \tag{A.4}$$

The representation of K^2 in the associated three-dimensional vectorspace $\mathcal{V}_3$ reads A_1, A_2, A_3, and A_4.

$$
\begin{aligned}
A_0 = {} & \frac{2804555520}{77}\, a_1\, a_2\, {a_3}^2 + \frac{605450371968}{77}\, {a_1}^2 a_2\, a_3 - \frac{135059124}{77}\, a_0\, a_2\, {a_3}^2 - \frac{6010}{77}\, a_0\, a_1\, {a_3}^2 \\
& - \frac{2800033978932}{77}\, a_0\, {a_1}^2 a_2 + \frac{134764032}{11}\, {a_0}^2 a_2\, a_3 - \frac{17180}{77}\, {a_0}^2 a_1\, a_3 + \frac{32036172480}{77}\, {a_0}^2 a_1\, a_2 \\
& + - \frac{714688}{77}\, a_0\, {a_1}^2 a_3 + \frac{31313904635520}{77}\, {a_1}^3 a_2 + \frac{4072590}{7}\, a_0\, {a_1}^3 + \frac{1354}{859} a_0 {a_1}^3 + \frac{14676}{7}\, {a_0}^3 a_1 \\
& - \frac{50149689261890868 7488}{77}\, a_0\, {a_2}^3 - \frac{491646}{7}\, {a_0}^2 {a_1}^2 + \frac{3436}{231}\, {a_0}^3 a_3 + \frac{370}{231}\, {a_0}^2 {a_3}^2 - \frac{44243964360}{77}\, a_0\, a_1\, a_2\, a_3 \\
& - \frac{2548431167119659984 53772288}{77}\, {a_2}^4 + \frac{3097783467920160}{77}\, {a_0}^2 {a_2}^2 + \frac{1236105325372290048}{77}\, {a_1}^2 {a_2}^2 \\
& + \frac{31512546733824}{77}\, {a_2}^2 {a_3}^2 - \frac{18583693388099568}{11}\, a_0\, {a_2}^2 a_1 - \frac{462376240397712}{77}\, a_0\, {a_2}^2 a_3 \\
& + \frac{8405757455996160}{77}\, a_1\, {a_2}^2 a_3 + \frac{62400}{77}\, {a_1}^2 {a_3}^2 + \frac{10720320}{77}\, {a_1}^3 a_3 + \frac{6241856322772562918 40}{77} a_1 {a_2}^3
\end{aligned}
$$

$$
\begin{aligned}
A_1 = {} & \frac{7795450836}{6013} a_1 a_2 {a_3}^2 + \frac{1854322773264}{6013} {a_1}^2 a_2 a_3 - \frac{249443640}{6013} a_0 a_2 {a_3}^2 + \frac{1877156016}{77}\, {a_0}^3 a_2 \\
& + \frac{277379217}{859} {a_0}^2 a_2 a_3 - \frac{230249}{12026} {a_0}^2 a_1 a_3 - \frac{607896774333}{6013} {a_0}^2 a_1 a_2 - \frac{23126}{6013} a_0 a_1 {a_3}^2 - \frac{931944}{6013} a_0 {a_1}^2 a_3 \\
& + \frac{9113015061180}{6013} {a_1}^3 a_2 + \frac{78113790}{6013} a_0 {a_1}^3 - \frac{3154799998762139520}{7} a_0 {a_2}^3 + \frac{5554463976}{7} a_0 {a_1}^2 a_2 \\
& - \frac{15868737}{12026} {a_0}^2 {a_1}^2 + \frac{8167}{12026} {a_0}^3 a_3 + \frac{370}{6013} {a_0}^2 {a_3}^2 - \frac{116131683888}{6013} a_0 a_1 a_2 a_3 + {a_0}^4 + \frac{21315870}{7} {a_0}^3 a_2 \\
& + \frac{183181}{12026} {a_0}^3 a_1 + \frac{203723929575236506176}{6013} {a_2}^3 a_3 + \frac{10458659095776}{7} {a_0}^2 {a_2}^2 + \frac{3208355030167295832}{6013} {a_1}^2 {a_2}^2 \\
& + \frac{42041979416520}{6013} {a_2}^2 {a_3}^2 - \frac{1722868108742016}{6013} a_0 {a_2}^2 a_3 + \frac{37555689842629920}{6013} a_1 {a_2}^2 a_3 + \frac{263640}{6013} {a_1}^2 {a_3}^2 \\
& + \frac{26100360}{6013} {a_1}^3 a_3 + \frac{2805836506613514268 1280}{6013} a_1 {a_2}^3 - \frac{50914502395604640}{859} a_0 {a_2}^2 a_1
\end{aligned}
$$

$$
\begin{aligned}
A_2 = {} & - \frac{91070}{9449} a_1 a_2 {a_3}^2 - \frac{80348216}{9449} {a_1}^2 a_2 a_3 + \frac{18214}{28347} a_0 a_2 {a_3}^2 + \frac{10}{85041} a_0 a_1 {a_3}^2 - \frac{13051633}{18898} {a_0}^2 a_1 a_2 \\
& + \frac{2344}{85041} a_0 {a_1}^2 a_3 + \frac{627758}{11} a_0 {a_1}^2 a_2 - \frac{1373879}{56694} {a_0}^2 a_2 a_3 - \frac{229}{340164} {a_0}^2 a_1 a_3 - \frac{6504439674}{9449} {a_1}^3 a_2 \\
& + \frac{103462675684992}{11} a_0 {a_2}^3 - \frac{97}{3436} {a_0}^2 {a_1}^2 - \frac{7}{1020492} {a_0}^3 a_3 - \frac{1}{255123} {a_0}^2 {a_3}^2 + \frac{8210084}{9449} a_0 a_1 a_2 a_3 \\
& - \frac{1013}{33} {a_0}^3 a_2 - \frac{7}{10308} {a_0}^3 a_1 + \frac{22775707244762784}{9449} {a_2}^3 a_3 + \frac{5597475820002317 7216}{11} {a_2}^4 - 15 {a_1}^4 \\
& - \frac{685728462}{11} {a_0}^2 {a_2}^2 - \frac{195539861692584}{9449} {a_1}^2 {a_2}^2 + \frac{480212818643684 5056}{77}\, {a_2}^3 a_3 \\
& + \frac{2161844880}{9449} {a_2}^2 {a_3}^2 + \frac{22760817374376}{9449} a_0 {a_2}^2 a_1 + \frac{16984148664}{9449} a_0 {a_2}^2 a_3 + \frac{347516067960}{9449} a_1 {a_2}^2 a_3 \\
& - \frac{25}{28347} {a_1}^2 {a_3}^2 - \frac{6770}{28347} {a_1}^3 a_3 - \frac{578350828631455776}{9449} a_1 {a_2}^3 - \frac{1031766224910256296235008}{7} {a_2}^4
\end{aligned}
$$

$$
\begin{aligned}
A_3 = {} & - \frac{19834131009888}{66143} a_1 a_2 {a_3}^2 - \frac{941190236611308}{66143} {a_1}^2 a_2 a_3 + \frac{644480670000}{66143} a_0 a_2 {a_3}^2 \\
& + \frac{1435760938}{66143} a_0 {a_1}^2 a_3 - \frac{121281245280}{77} a_0 {a_1}^2 a_2 + \frac{290340962025}{9449} {a_0}^2 a_2 a_3 + \frac{20482048}{66143} a_0 a_1 {a_3}^2 \\
& - \frac{18124121}{132286} {a_0}^2 a_1 a_3 - \frac{13898071217925}{66143} {a_0}^2 a_1 a_2 + 2 a_0 {a_3}^3 + \frac{8672396439985200}{66143} {a_1}^3 a_2 \\
& - \frac{17177378585369109120}{77} a_0 {a_2}^3 - \frac{347839545}{12026} {a_0}^2 {a_1}^2 + \frac{4549933}{396858} {a_0}^3 a_3 + \frac{967916}{198429} {a_0}^2 {a_3}^2 \\
& - \frac{5398282254528}{66143} a_0 a_1 a_2 a_3 + \frac{194360970}{77} {a_0}^3 a_2 + \frac{6145575}{12026} {a_0}^3 a_1 - \frac{785889483178772038367808}{66143} {a_2}^3 a_3 \\
& - 674172 a_2 {a_3}^3 - 30 a_1 {a_3}^3 - \frac{2091697646230119566 4261120}{77} {a_2}^4 + \frac{169526691419220}{77} {a_0}^2 {a_2}^2 \\
& + \frac{4465185813593590924 80}{66143} {a_1}^2 {a_2}^2 - \frac{253905865676196912}{66143} {a_2}^2 {a_3}^2 - \frac{2829113959411116000}{9449} a_0 {a_2}^2 a_1 \\
& + \frac{1560166342628932800}{66143} a_0 {a_2}^2 a_3 - \frac{44983252019780640144}{66143} a_1 {a_2}^2 a_3 - \frac{379824420}{66143} {a_1}^2 {a_3}^2 \\
& - \frac{22056787770}{66143} {a_1}^3 a_3 + \frac{485516352640069936542 2400}{66143} a_1 {a_2}^3 + \frac{1917419400}{6013} a_0 {a_1}^3.
\end{aligned}
$$

This (homogeneous) *quartic* mapping is a birational mapping and is *integrable*: it yields *elliptic curves*. It corresponds to a representation of the birational transformation K^2 in four homogeneous variables. They are an infinite number of such integrable mappings: as many as initial matrices M_0.

References

1. H.E. Lieb, F.Y. Wu, in *Phase transition and Critical Phenomena*, edited by C. Domb, M.S. Green (Academic Press, 1972) Vol. 1.
2. Y.G. Stroganov, Phys. Lett. A **74**, 116 (1979).
3. S. Boukraa, J.-M. Maillard, G. Rollet, *Determinantal identities on integrable mappings*, Int. J. Mod. Phys. B **8**, 2157–2201 (1994).
4. S. Boukraa, J.-M. Maillard, G. Rollet, Physica A **208**, 115–175 (1994).
5. S. Boukraa, J.-M. Maillard, G. Rollet, Int. J. Mod. Phys. B **8**, 137–174 (1994).
6. M.P. Bellon, J.-M. Maillard, C-M. Viallet, Phys. Lett. A **159**, 221–232 (1991).
7. M.P. Bellon, J.-M. Maillard, C.-M. Viallet, Phys. Lett. A **159**, 233–244 (1991).
8. M.P. Bellon, J.-M. Maillard, C.-M. Viallet, Phys. Lett. A **157**, 343–353 (1991).
9. M.P. Bellon, J.-M. Maillard, C.-M. Viallet, Phys. Lett. B **260**, 87–100 (1991).
10. R.J. Baxter, *Exactly solved models in statistical mechanics* (London Acad. Press, 1981).
11. R.J. Baxter, Ann. Phys. **76**, 25–47 (1973).
12. M.P. Bellon, J.-M. Maillard, C.-M. Viallet, Phys. Rev. Lett. **67**, 1373–1376 (1991).
13. M.P. Bellon, J.-M. Maillard, C.-M. Viallet, Phys. Lett. B **281**, 315–319 (1992).
14. A. Gaaf, J. Hijmans, Physica A **80**, 149-171 (1975).
15. C.M. Viallet, G. Falqui, Comm. Math. Phys. **154**, 111–125 (1993).
16. S. Boukraa, J.-M. Maillard, G. Rollet, J. Stat. Phys. **78**, 1195–1251 (1995).
17. J.-M. Maillard, J. Math. Phys. **27**, 2776 (1986).
18. S. Boukraa, J.-M. Maillard, Physica A **220**, 403–470 (1995).
19. S. Boukraa, J.-M. Maillard, J. Phys. I France **3**, 239–258 (1993).
20. J.-M. Maillard, Rollet, J. Phys. A **27**, 6963–6986 (1994).
21. anonymous@crtbt.polycnrs-gre.fr